# Complexity of Rectangular Mazes

**1. Introduction.** Consider the mazes in figure 1. Of these two mazes, which one is the more challenging? If we can order these mazes according to this criterion, then the possibility of such an ordering of all mazes seems plausible. Given two mazes, it is plausible that the maze that is more complex will probably be the one that is most difficult to traverse from entrance to exit. In this short paper we will develop a method by which the complexity of a certain type of maze can be measured.

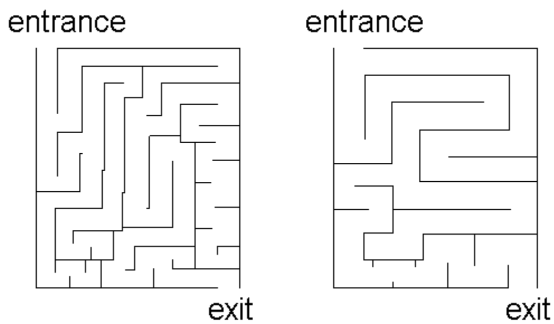


figure 1

Basically, there are two types of mazes: rectangular and non-rectangular. A maze is rectangular if and only if every wall of the maze is a straight line and if every two walls that meet in the maze do so at a right angle. The mazes in figure 1 are rectangular mazes and the maze in figure 2 is a non-rectangular maze. This paper will describe how to measure the complexity of rectangular mazes. To measure the complexity of non-rectangular mazes, see [**1**].

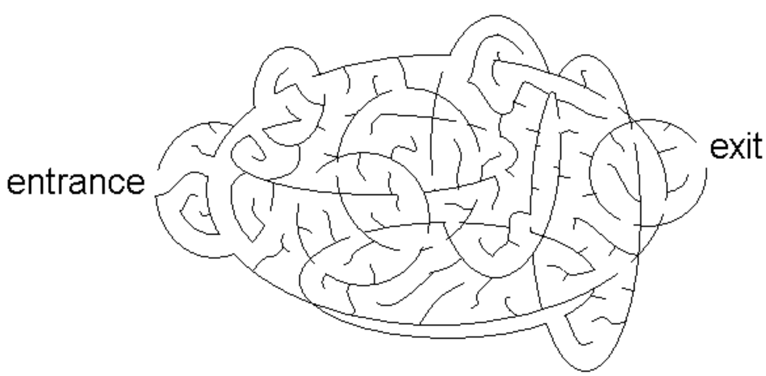


figure 2

**2. The Graph of a Maze.** A graph is a continuum that can be written as the union of finitely many arcs any two of which are either disjoint or intersect only in one or both of their endpoints [**2**]. Let *M* be a maze. We want to associate *M* with a connected graph. Every hallway of *M* has two walls with a pathway between them. The graph of a maze can be found by tracing lines down the middle of each hallway of the maze in a manner so that the total length of all traced lines is minimal. See figure 3.

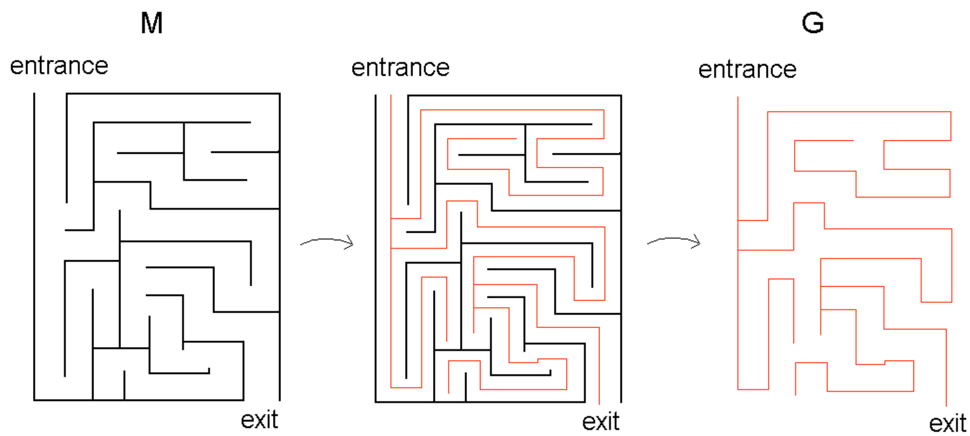


figure 3

Suppose that a maze *M* has the graph *G*. We will assume that *G* contains two points, called the *gates*, which are distinguished from all other points in the graph. One gate will be the *entrance* to the graph and the other gate will be the *exit* of the graph. We denote the order of a point $x \in G$ by $ord(x)$. If *e* is a point in *G* other than one of the gates such that $\mathrm{ord}(e) = 1$, then *e* is called a *dead-end* of *G*. See figure 4.

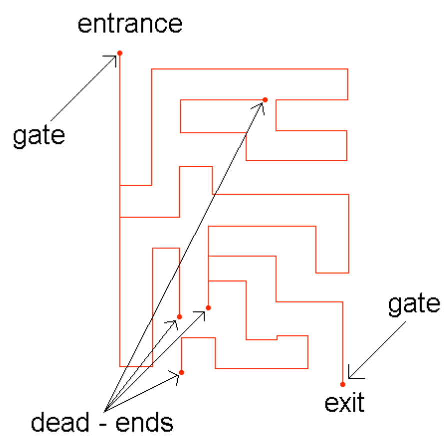


figure 4

Now consider the subset *H* of *G* defined by $H = \{ p \in G \,|\, \mathrm{ord}(p) \le 2 \}$. Each component *h* of *H* is called a *hallway*. Every point not in *H* is called an intersection. Every intersection in *G* represents a location in *M* where there is a choice of different paths to follow when attempting to traverse the maze from entrance to exit. If *u* and *v* are intersections, gates or dead-ends such that $\{u\} \cup h \cup \{v\}$ is a connected set then we call *u* and *v* the *endpoints* of *h*. See figure 5.

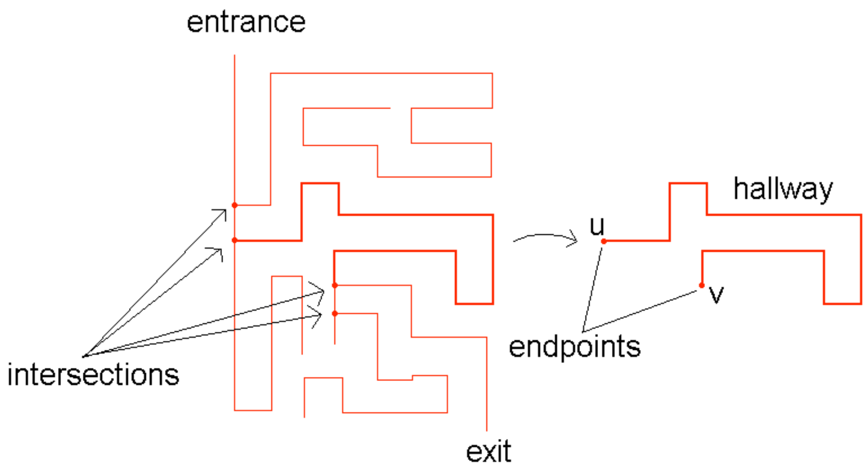

figure 5

**3. The Complexity Measure of a Hallway.** Consider the three hallways in figure 6. As it is possible to order these hallways by how "complicated" or "confusing" each of these hallways are, we want to develop a method by which we could order all hallways accordingly. Given two mazes, we say the more "complicated" of the two has a higher measure of complexity. It appears that the more quickly a hallway alters its direction the higher the measure of its complexity, or the more complex the hallway becomes. Therefore, the measure of the complexity of a hallway will depend on how "quickly" it changes direction.

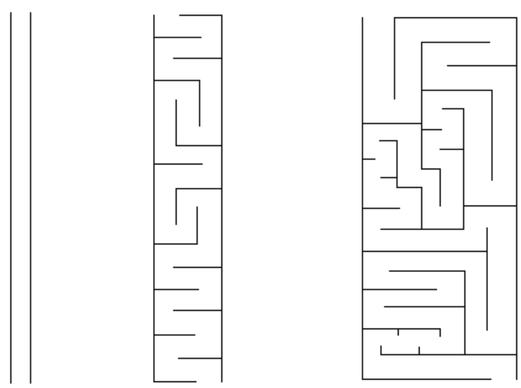
figure 6

Let $M$ be a rectangular maze with graph $G$. Let $h$ be a hallway in $M$ with total length equal to $D$. Note that whenever $h$ in $M$ changes direction, the graph of $h$ in G will change direction by $90^\circ$. We call a point where the graph changes direction by $90^\circ$ a *corner point* of $h$. Therefore, the only variable involved in determining the complexity of $h$ will be the length of each subset of $h$ between its corner points. The shorter each such subset is, the more complex $h$ is. So, let $w_1, w_2, \ldots, w_n$ be the corner points of the hallway $h$, and denote the length of the subset of $h$ between $w_{i-1}$ and $w_i$ by $d(w_i)$. The complexity of $h$ increases as each value of $\dfrac{1}{d(w_i)}$ increases. Also, the larger $D$ is the more complex $h$ is. For these reasons, we have the following definition.

**Definition 1 (The Complexity of a Hallway).** Let $h$ be a hallway of total length $D$ in a rectangular maze $M$ that has corner points $w_1, w_2, \ldots, w_n$ in its graph $G$. The *complexity* of $h$, denoted $\gamma(h)$ is defined by

$$\gamma(h) = D\sum_{i=1}^{n} \frac{1}{d(w_i)}.$$

**4. The Complexity Measure of a Maze.** As the total complexity of a maze is the total complexity of all of its hallways, we arrive at the following definition.

**Definition 2 (The Complexity of a Maze).** Let $M$ be a maze with $m$ total hallways and suppose each hallway $h_k$ in $M$ has complexity $\gamma(h_k)$. The *complexity C* of $M$ is then defined by

$$C = \sum_{k=1}^{m} \gamma(h_k).$$

Note that since the units of distance appear in both the numerator and the denominator, then $\gamma(h)$ is unit-less. That is, $C$ will be independent of the units of measurement of length.

For example, consider the maze from figure 3 shown below in figure 7.

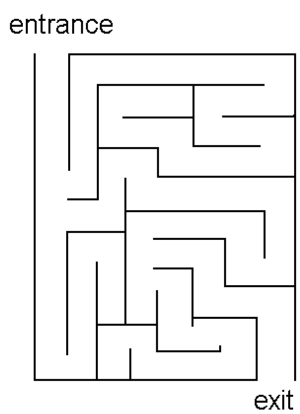


figure 7

Its graph is given in figure 8.

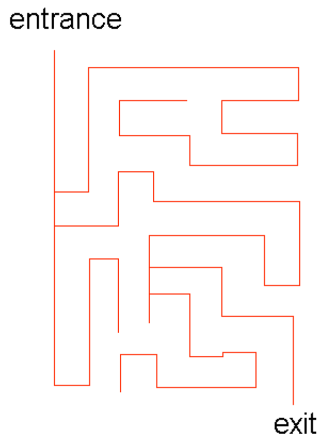


figure 8

There are 2 gates, 4 dead-ends, 4 intersections, 8 hallways and 37 corner points. The 8 hallways, their lengths and the distances between each of their corner points are shown in figure 9.

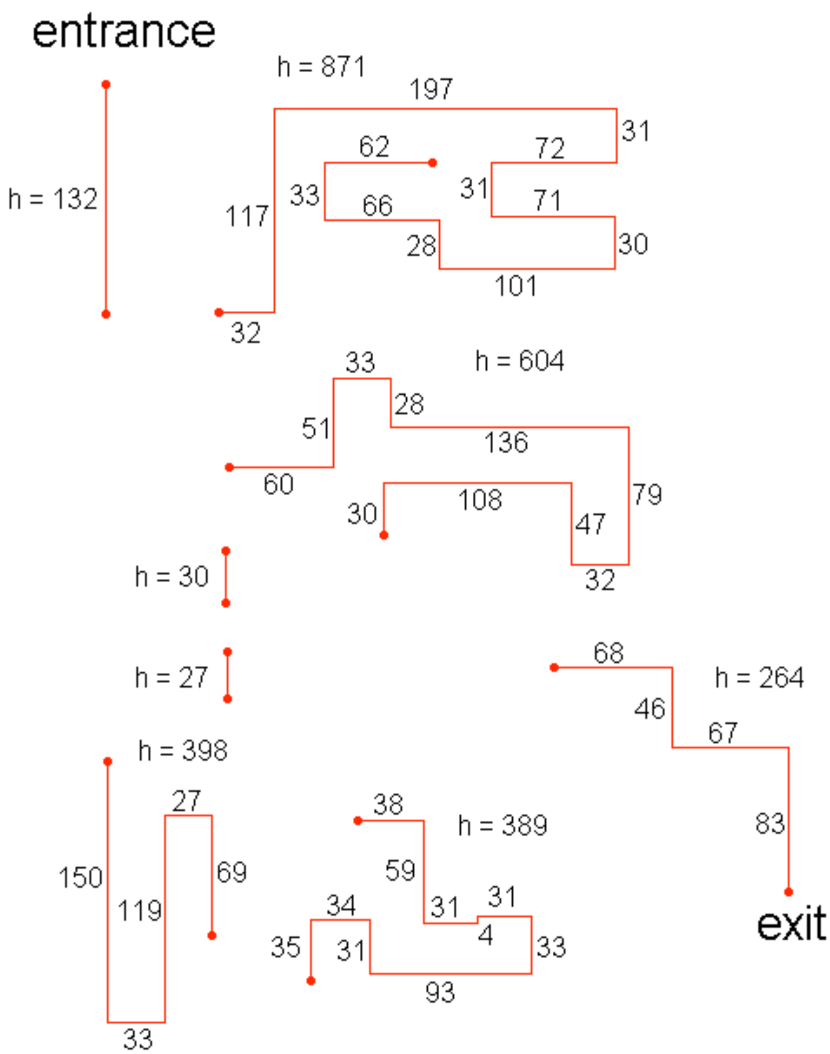


figure 9

Now we use Definition 1 to calculate the complexity of each hallway.

$$\gamma(h_1) = D\sum_{i=1}^{n}\frac{1}{d(w_i)} = 132\left(\frac{1}{132}\right) = 1$$

$$\gamma(h_2) = 871\left(\frac{1}{32} + \cdots + \frac{1}{66} + \frac{1}{33} + \frac{1}{62}\right) = 242.05$$

$$\gamma(h_3) = 604\left(\frac{1}{60} + \cdots + \frac{1}{47} + \frac{1}{108} + \frac{1}{30}\right) = 131.32$$

$$\gamma(h_4) = 30\left(\frac{1}{30}\right) = 1$$

$$\gamma(h_5) = 27\left(\frac{1}{27}\right) = 1$$

$$\gamma(h_6) = 398\left(\frac{1}{150} + \frac{1}{33} + \frac{1}{119} + \frac{1}{27} + \frac{1}{69}\right) = 38.57$$

$$\gamma(h_7) = 389\left(\frac{1}{38} + \cdots + \frac{1}{31} + \frac{1}{34} + \frac{1}{35}\right) = 190.25$$

$$\gamma(h_8) = 264\left(\frac{1}{68} + \frac{1}{46} + \frac{1}{67} + \frac{1}{83}\right) = 16.74$$

Finally, we use Definition 2 to calculate the complexity of the maze.

$$C = \sum_{k=1}^{8}\gamma(h_k) = 1 + 242.05 + 131.32 + 1 + 1 + 38.57 + 190.25 + 16.74 = 621.93\,.$$

As a final comparative note we calculate the complexity of the two mazes in figure 1 to be 1162.58 and 435.66, respectively.

## Bibliography


[**1**] M. McClendon, *The Complexity and Difficulty of a Maze*, Bridges: Mathematical Connections in Art, Music, and Science, Conference Proceedings 2001, pp. 213 – 222.
[**2**] J. Munkres, *Topology*, *A First Course*, Prentice Hall, New Jersey, 1975.
[**3**] S. Nadler, *Continuum Theory*, *An Introduction*, Marcel Dekker, New York, 1992.
[**4**] P. Anthony, “The Topology of Roman Mosaic Mazes”, in *The Visual Mind: Art and Mathematics*, edited by M. Emmer, MIT Press, 1994, pp. 65 – 73.